

On the generalized dimensions of multifractal eigenstates

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Recently [Europhys. Lett. **98**, 37006 (2012)], based on heuristic arguments, it was conjectured that an intimate relation exists between the eigenfunction multifractal dimensions D_q of the eigenstates of critical random matrix ensembles $D_{q'} \approx qD_q[q' + (q - q')D_q]^{-1}$, $1 \leq q \leq 2$. Here, we verify this relation by extensive numerical calculations on critical random matrix ensembles and extend its applicability to $q < 1/2$ and also to deterministic models producing multifractal eigenstates. We also demonstrate, for the scattering version of the power-law banded random matrix model at criticality, that the scaling exponents σ_q of the inverse moments of Wigner delay times, $\langle \tau_w^{-q} \rangle \propto N^{-\sigma_q}$ where N is the linear size of the system, are related to the level compressibility χ as $\sigma_q \approx q(1 - \chi)[1 + q\chi]^{-1}$ for a limited range of q ; thus providing a way to probe level correlations by means of scattering experiments.

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I. INTRODUCTION

It is well-known that the spatial fluctuations of the eigenstates in a disordered system at the Anderson-transition show multifractal characteristics^{1,2} which has been demonstrated recently in a series of experiments.³ Therefore the modeling and analysis of multifractal states has become of central importance producing many interesting results. For this purpose random matrix models have been invoked and studied recently.⁴⁻⁷

Since the exact, analytical prediction of the multifractal dimensions of the states for the experimentally relevant Anderson-transition in $d = 3$ or the integer quantum-Hall transition in $d = 2$ seems to be out of reach, it is desirable to search for heuristic relations in order to understand the complexity of the states at criticality. In the present paper we propose such heuristic relations that are numerically verified using various ensembles of random matrices. Moreover, we will demonstrate the robustness of these findings by extending to many different observables. These relations could play an important role in testing analytical predictions and could also be tested using the experiments mentioned above.

The spatial fluctuations of the eigenstates can be described by a set of multifractal dimensions D_q defined by the scaling of the inverse mean eigenfunction participation numbers as a function of system size N :

$$\left\langle \sum_{i=1}^N |\Psi_i|^{2q} \right\rangle \sim N^{-(q-1)D_q}, \quad (1)$$

where $\langle \dots \rangle$ is the average over some eigenvalue window and over random realizations of the matrix. For strongly localized eigenstates these quantities do not scale with system size, i.e. $D_q \rightarrow 0$ for all q , while extended states always feel the entire system, i.e. $D_q \rightarrow d$, for all q . Mul-

tifractal states, on the other hand, should be described by the series of the D_q , which are a nonlinear function of the parameter q .

One of the most important generalized dimensions often used in this context is the information dimension D_1 . It is defined through the scaling of the mean eigenfunction entropy with the logarithm of the system size:

$$\left\langle - \sum_{i=1}^N |\Psi_i|^2 \ln |\Psi_i|^2 \right\rangle \sim D_1 \ln N. \quad (2)$$

A further, well-known and widely used dimension is called the correlation dimension D_2 , which is extracted from the inverse participation number from Eq. (1) using $q = 2$. The correlation dimension D_2 shows up in the power-law scaling of the density-density correlation function in the energy domain as⁸

$$\langle |\Psi_\mu|^2 |\Psi_\nu|^2 \rangle \propto \left| \frac{E_0}{E_\mu - E_\nu} \right|^{1-D_2/d}, \quad (3)$$

as well as in the auto-correlation in space as⁹

$$\langle |\Psi_k|^2 |\Psi_l|^2 \rangle \propto \left(\frac{N}{|\mathbf{r}_k - \mathbf{r}_l|} \right)^{d-D_2}, \quad (4)$$

where \mathbf{r}_k and \mathbf{r}_l denote the position of sites k and l , respectively. In Eq. (3) the energy scale E_0 is of the order of the bandwidth.⁸

Besides the wave functions the spectra also show anomalous properties. Spectral fluctuations can be characterized in many ways. A usual, often employed quantity is the level compressibility χ , which is extracted from the asymptotic behavior of the spectral number variance as

$$\Sigma^{(2)}(E) = \langle n(E)^2 \rangle - \langle n(E) \rangle^2 \sim \chi E, \quad (5)$$

where $n(E)$ stands for the number of states in an interval of length E if $E \gg 1$. The spectral fluctuations in a metallic system with extended states yield a vanishing compressibility, $\chi \rightarrow 0$, while in a strongly disordered insulating system the levels are uncorrelated, hence they are easily compressible, $\chi = 1$. However, for the multifractal states an intermediate statistics exists, $0 < \chi < 1$, furthermore the spectral and eigenstate statistics are supposed to be coupled, which has been pointed out first in Ref. [10].

In a recent work⁴ Bogomolny and Giraud have shown that in a d -dimensional critical system the information dimension D_1 and the level compressibility χ are related in a simple way as

$$\chi + D_1/d = 1, \quad (6)$$

furthermore the generalized dimensions D_q can be expressed as

$$\frac{D_q}{d} = \begin{cases} \frac{\Gamma(q-1/2)}{\sqrt{\pi}\Gamma(q)}(1-\chi), & 1-\chi \ll 1 \\ 1-q\chi, & \chi \ll 1 \end{cases}. \quad (7)$$

These expressions have been shown to be valid for various critical random matrix ensembles in Ref. [4].

As for the critical, three-dimensional Anderson transition and the two-dimensional quantum-Hall transition it has been shown earlier that there exists another relation between the level compressibility χ and the correlation dimension D_2 :¹⁰

$$2\chi + D_2/d = 1. \quad (8)$$

This relation should obviously hold approximately only since the range of the generalized dimension and that of the level compressibility are limited as $0 \leq D_2/d \leq 1$ but $0 \leq \chi \leq 1$, leaving the range of validity for the limit of weak-multifractality.

Looking at Eqs. (6) and (8) it seems evident that there may exist further relations between generalized dimensions and the level compressibility, moreover, one may conjecture that the generalized dimensions of the multifractal spectrum must be also intimately linked together.

In the present work we show a series of relations between various generalized dimensions, D_q and $D_{q'}$, and the level compressibility χ allowing for a generalization that for particular cases yields Eq. (6) exactly and Eq. (8) in the appropriate limit. In order to prove that, numerical simulations of various critical random matrix ensembles will be used. Moreover, by the use of our relations between generalized dimensions we state a clear link between the spectral and scattering properties of disordered systems at the metal-insulator transition. Finally, by exploring a deterministic model having a self-similar potential that produces multifractal eigenstates, we also show that our results are not restricted to random matrix models.

II. MODEL AND HEURISTIC RELATIONS

In Ref. [4] Eqs. (6) and (7) were shown to be correct numerically for the Power-Law Banded Random Matrix (PBRM) model^{2,11,12} at criticality. Below we will make use of this model to derive our main results.

The PBRM model describes one-dimensional (1d) samples of length N with random long-range hoppings. This model is represented by $N \times N$ real symmetric ($\beta = 1$) or complex hermitian ($\beta = 2$) matrices whose elements are statistically independent random variables drawn from a normal distribution with zero mean and a variance given by $\langle |H_{mn}|^2 \rangle = \beta^{-1}$ and

$$\langle |H_{mn}|^2 \rangle = \frac{1}{2} \frac{1}{1 + [\sin(\pi|m-n|/N)/(\pi b/N)]^{2\mu}}, \quad (9)$$

where b and μ are parameters. In Eq. (9) the PBRM model is in its periodic version; i.e. the 1d sample is in a ring geometry. Theoretical considerations^{2,11-13} and detailed numerical investigations^{2,14,15} have verified that the PBRM model undergoes a transition at $\mu = 1$ from localized states for $\mu > 1$ to delocalized states for $\mu < 1$. This transition shows key features of the disorder driven Anderson metal-insulator transition,² including multifractality of eigenfunctions and non-trivial spectral statistics. Thus the PBRM model possesses a line of critical points $b \in (0, \infty)$ in the case of $\mu = 1$. In the following we will focus on the PBRM model at criticality, $\mu = 1$. By tuning the parameter b , from $b \ll 1$ to $b \gg 1$, the states cross over from the nature of strong-multifractality ($D_q \rightarrow 0$) which corresponds to localized-like or insulator-like states to weak-multifractality ($D_q \rightarrow 1$) showing rather extended, i.e. metallic-like states. Meanwhile at the true, Anderson transition in $d = 3$ or at the integer quantum-Hall transition in $d = 2$, the states belong to the weakly multifractal regime, i.e. $d - D_2 \ll d$, the PBRM model allows for an investigation without such a limitation. The evolution of the generalized dimensions as a function of the parameter b therefore represent this behavior, i.e. $D_q \rightarrow 1$ for $b \gg 1$ and in the other limit of $b \ll 1$ the multifractal dimensions vanish as $D_q \sim b^{2,12}$.

The multifractal dimensions, especially D_1 and D_2 and their dependence on the parameter b in the case of the PBRM model, have been at the focus of several of our works.^{16,17} In those works it has been demonstrated that a simple, phenomenological relation can be identified: $D_1 \approx [1 + (\alpha_1 b)^{-1}]^{-1}$ and $D_2 \approx [1 + (\alpha_2 b)^{-1}]^{-1}$ where $\alpha_{1,2}$ are fitting constants. These continuous functions are trivial interpolations between the limiting cases of low- b and large- b , taking the half of the harmonic mean of the two cases. In Ref. [18] we generalized and propose the following heuristic expression for an extended range of the parameter q

$$D_q \approx [1 + (\alpha_q b)^{-1}]^{-1}, \quad (10)$$

as a global fit for the multifractal dimensions D_q of the PBRM model for both symmetry classes, $\beta = 1$ and

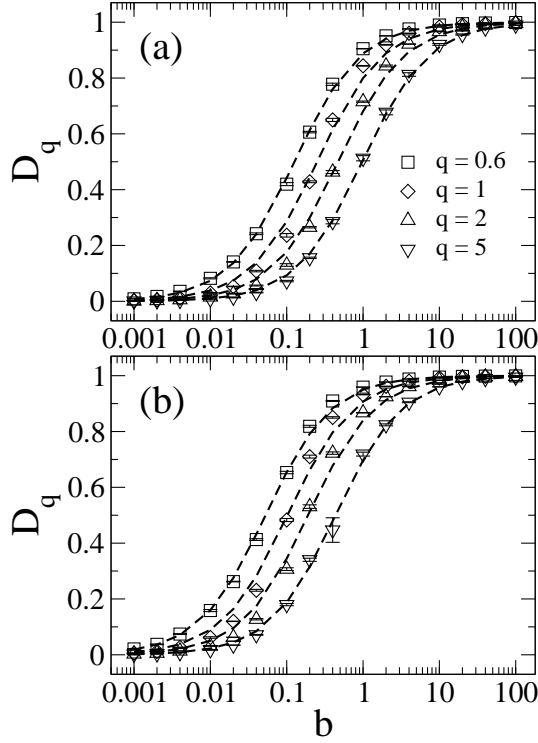


FIG. 1: D_q as a function of b for the PBRM model at criticality with (a) $\beta = 1$ and (b) $\beta = 2$. The dashed lines are fits of the numerical data with Eq. (10).

$\beta = 2$. In Fig. 1 we show fits of Eq. (10) to numerically obtained D_q 's as a function of b for some values of q and in Fig. 2(a) we plot the values of α_q extracted from the fittings. We observe that Eq. (10) fits reasonably well the numerical D_q for $q > 1/2$. It is important to stress that Eq. (10) reproduces well the b -dependencies predicted analytically² for the limits $b \ll 1$ and $b \gg 1$.

The multifractal dimensions of Fig. 1 were extracted from the linear fit of the logarithm of the inverse mean eigenfunction participation numbers versus the logarithm of N , see Eq. (1). D_1 was extracted from the linear fit of the mean eigenfunction entropy versus the logarithm of N , see Eq. (2). We used system sizes of $N = 2^n$ with $8 \leq n \leq 13$. The average was performed over 2^{n-3} eigenvectors with eigenvalues around the band center of 2^{16-n} realizations of the random matrices.

By inspecting the obtained fittings using Eq. (10) one may notice that Eq. (6) leads to

$$\chi \approx (1 + \alpha_1 b)^{-1}, \quad (11)$$

which also reproduces qualitatively well the b -dependencies predicted analytically^{2,4} in the small-

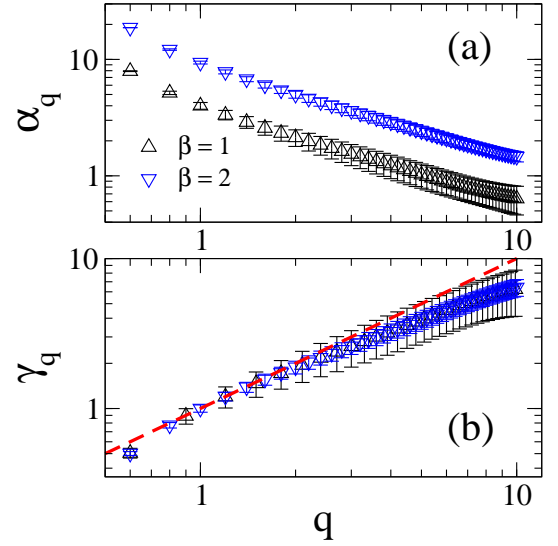


FIG. 2: (Color online) (a) α_q and (b) $\gamma_q = \alpha_1/\alpha_q$ as a function of q for the PBRM model at criticality with $\beta = 1$ and $\beta = 2$. The red dashed line in (b) equal to q is plotted to guide the eye. The error bars in (a) are the rms error of the fittings.

and large- b limits:

$$\chi = \begin{cases} 1 - 4b, & \beta = 1 \\ 1 - \pi\sqrt{2}b - 4\left(\frac{2}{\sqrt{3}} - 1\right)\pi^2 b^2, & \beta = 2 \\ \frac{1}{2\beta\pi b} & \end{cases} \quad \begin{matrix} b \ll 1 \\ b \gg 1 \end{matrix}. \quad (12)$$

As a consequence of equating b in Eqs. (10) and (11) we get a direct relation

$$\chi \approx (1 - D_q)[1 + (\gamma_q - 1)D_q]^{-1}, \quad (13)$$

with $\gamma_q = \alpha_1/\alpha_q$. We observed that $\gamma_q \approx q$ in the range $0.8 < q < 2.5$, as plotted in Fig. 2(b), so in this range of q values we can write simplified relations between χ and D_q :

$$\chi \approx \frac{1 - D_q}{1 + (q - 1)D_q} \quad \text{and} \quad D_q \approx \frac{1 - \chi}{1 + (q - 1)\chi}. \quad (14)$$

The expression for D_q in Eq. (14) reproduces Eq. (7) exactly for $q = 1$ and $q = 2$ and approximately for $1 < q < 2.5$. Moreover, Eq. (14) combined with Eq. (6) allows us to express any D_q in terms of, for example, D_1 :

$$D_q \approx D_1 [q + (1 - q)D_1]^{-1}. \quad (15)$$

We also noticed that by equating χ for different D_q 's from Eq. (14) we could get recursive relations among them:

$$q'D_{q'}(1 - D_{q'})^{-1} = qD_q(1 - D_q)^{-1}, \quad (16)$$

which in case of taking $q' = q + 1$ leads to

$$D_{q+1} = qD_q(1 + q - D_q)^{-1}. \quad (17)$$

Notice that all these relations can be expressed using the fact that the ratio $qD_q/(1 - D_q)$ is independent of q . These expressions also provide a relation between the correlation dimension and the information dimension or between the correlation dimension and the compressibility of the spectrum:

$$D_2 = D_1 (2 - D_1)^{-1} = (1 - \chi) (1 + \chi)^{-1} . \quad (18)$$

It is relevant to add that in the weak multifractal regime, i.e. when $\chi \rightarrow 1$, Eq. (18) reproduces the relation given in Eq. (8) with $d = 1$, reported in [10].

A. The case $q < 1/2$

For $q < 1/2$, Eq. (10) cannot be directly applied. However, the regime $q < 1/2$ could also be explored within our approach by the combination of Eq. (10) and the symmetry relation¹⁹

$$\Delta_q = \Delta_{1-q} \quad \text{with} \quad \Delta_q = D_q(q - 1) - d(q - 1) , \quad (19)$$

implying that it is possible to link the multifractal dimensions with indexes $q < 1/2$ to those with $q > 1/2$. Therefore, we get

$$D_q \approx \frac{1 - 2q}{1 - q} + \frac{q}{1 - q} \left(\frac{\alpha_{1-q} b}{1 + \alpha_{1-q} b} \right) , \quad (20)$$

for $q < 1/2$; that is, once we know the coefficients α_q for $q > 1/2$ we can use them to get D_q for $q < 1/2$. Moreover, by the use of Eq. (17) it is possible to write D_q , for $q < 1/2$, as a function of any specific D_q with $q > 1/2$. For example,

$$D_q \approx \frac{1 - 2q}{1 - q} + \frac{q}{1 - q} \left(\frac{D_1}{1 + q(D_1 - 1)} \right) , \quad (21)$$

provides D_q for $q < 1/2$ in terms of the information dimension. Moreover, we can write down relations between χ and D_q with $q < 1/2$:

$$\chi \approx \frac{1 - D_q}{q(2 - D_q)} \quad \text{and} \quad D_q \approx \frac{1 - 2q\chi}{1 - q\chi} . \quad (22)$$

Finally, it is important to stress that Eqs. (20-21) get the form

$$D_q = (1 - 2q)(1 - q)^{-1} \quad (23)$$

in the limit $b \rightarrow 0$, which has been derived analytically in Refs. [6,19-21].

B. Wigner delay times

As mentioned in the Introduction, the modeling and analysis of multifractal states in disordered systems at the Anderson-transition has been a subject of intensive

research activity for many decades.^{1,2} Moreover, since the properties of the closed system, i.e. the fractality of the eigenstates, strongly influence the scattering and transport properties of the corresponding open system, the interest has also been extended to critical *scattering* systems. In particular, much attention has been focused on the probability distribution functions of the resonance widths and Wigner delay times,^{17,19,22-30} as well as the transmission or dimensionless conductance.³¹⁻⁴⁷

Among many relevant results, here we want to focus on Wigner delay times τ_w and recall that

- (i) for disordered systems at criticality the inverse moments of Wigner delay times $\langle \tau_w^{-q} \rangle$ scale as^{19,22,23}

$$\langle \tau_w^{-q} \rangle \propto N^{-\sigma_q} \quad \text{where} \quad \sigma_q \equiv qD_{q+1} ; \quad (24)$$

- (ii) for the PBRM model at criticality the typical values of the Wigner delay times scale as¹⁷

$$\tau_w^{\text{typ}} \propto N^{\sigma_\tau} \quad \text{where} \quad \sigma_\tau = D_1 , \quad (25)$$

with $\tau_w^{\text{typ}} \equiv \exp \langle \ln \tau_w \rangle$.

Note that Eqs. (24) and (25) provide a way to probe the properties of a critical system (i.e. the fractality of its eigenstates) by means of scattering experiments. Moreover, we can also relate spectral properties to scattering properties by

- (i) combining Eqs. (14) and (24):

$$\sigma_q \approx \frac{q(1 - \chi)}{1 + q\chi} \quad \text{where} \quad \chi \approx \frac{q - \sigma_q}{q(\sigma_q + 1)} ; \quad (26)$$

- (ii) combining Eqs. (18) and (25):

$$\sigma_\tau = 1 - \chi . \quad (27)$$

Also, we can express any σ_q as a function of, say, σ_1 :

$$\sigma_q = 2q\sigma_1[(1 + q) + (1 - q)\sigma_1]^{-1} . \quad (28)$$

Finally, note that we can obtain recursive relations for σ_q 's in analogy to Eq. (16):

$$(q' + 1)\sigma_{q'}(q' - \sigma_{q'})^{-1} = (q + 1)\sigma_q(q - \sigma_q)^{-1} , \quad (29)$$

or

$$\sigma_{q'} = q'(q + 1)\sigma_q[q(q' + 1) + (q - q')\sigma_q]^{-1} , \quad (30)$$

which leads to

$$\sigma_{q+1} = (q + 1)^2 \sigma_q [q(q + 2) - \sigma_q]^{-1} , \quad (31)$$

when $q' = q + 1$.

III. NUMERICAL RESULTS FOR THE PBRM MODEL

In this section we present numerical justification of our analytical relations derived above using the PBRM model.

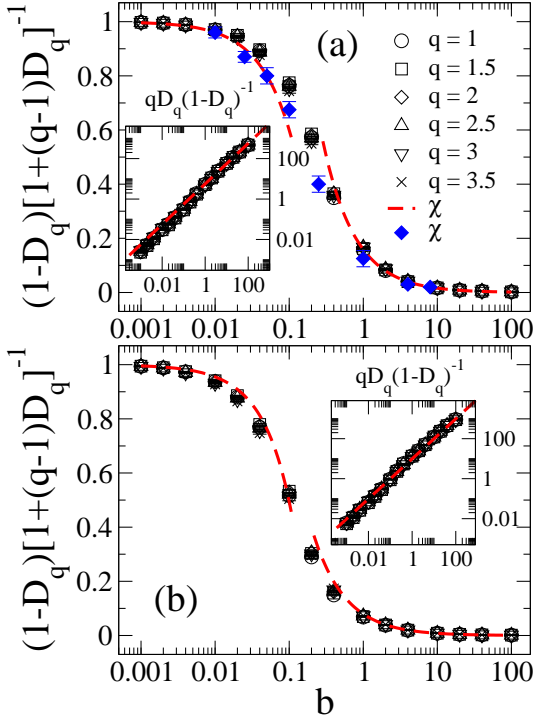


FIG. 3: (Color online) $(1 - D_q)[1 + (q - 1)D_q]^{-1}$ [see Eq. (14)] as a function of b for the PBRM model at criticality with (a) $\beta = 1$ and (b) $\beta = 2$. The red dashed lines are the analytical predictions for χ given by Eq. (12). The blue symbols in (a) are independent numerically obtained values of χ (taken from Ref. [48]). Insets: $qD_q(1 - D_q)^{-1}$ as a function of b ; see Eq. (32). The red dashed line equal to $\alpha_1 b$ is plotted to guide the eye. (a) $\alpha_1 = 4$ and (b) $\alpha_1 = 9.4$ were used.

A. Multifractal exponents

In Fig. 3 we plot $(1 - D_q)[1 + (q - 1)D_q]^{-1}$ as a function of b for several values of q for the PBRM model at criticality with $\beta = 1$ and $\beta = 2$ and observe good correspondence with the analytical prediction for χ ; that is, we verify the validity of Eq. (14). For completeness in Fig. 3(a) we also include independent numerically obtained values of χ (taken from Ref. [48]). In the insets of Fig. 3 we plot $qD_q(1 - D_q)^{-1}$ as a function of b , see Eq. (16), which for the PBRM model acquires the simple form

$$qD_q(1 - D_q)^{-1} \approx \alpha_1 b. \quad (32)$$

The fact that all curves $qD_q(1 - D_q)^{-1}$ vs. b fall one on top of the other makes evident its independence of q .

In Fig. 4 we present D_q as a function of q for the PBRM model with $\beta = 1$ and $\beta = 2$ for some values of b . As black and red dashed lines we also include Eqs. (15), for $q > 1/2$, and (21), for $q < 1/2$, respectively. In both equations we used values of D_1 we have obtained numerically. We observe very good correspondence between the numerical data and Eqs. (21) and (15) mainly for $-4 < q < 0.2$ and $0.8 < q < 4$, respectively.

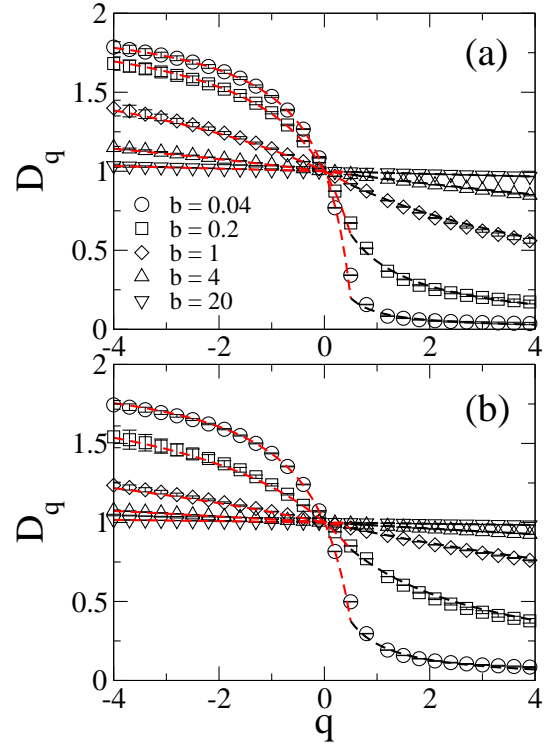


FIG. 4: D_q as a function of q for the PBRM model at criticality with (a) $\beta = 1$ and (b) $\beta = 2$. Black [Red] dashed lines are Eq. (15) [Eq. (21)]. We used (a) [(b)] $D_1 = 0.109, 0.429, 0.843, 0.958$, and 0.991 [$D_1 = 0.230, 0.711, 0.931, 0.980$, and 0.996].

B. Wigner delay times

We obtain Wigner delay times τ_w by turning the isolated system, represented by the PBRM model, into a scattering one by attaching one semi-infinite single-channel lead using perfect coupling. Since we are dealing here with the periodic version of the PBRM model, all sites are bulk sites and the place at which we attach the lead is irrelevant. To compute τ_w we use the *effective Hamiltonian approach* described in Refs. [17,22]. For statistical processing a large number of disorder realizations is used. Each disorder realization gives one value of τ_w . We used $N = 50, 100, 200, 400$, and 800 getting $10^6, 10^6, 10^5, 10^5$, and 10^4 values of τ_w , respectively. Then, the exponents σ_q [σ_τ] were extracted from the linear fit of the logarithm of the averaged inverse moments of Wigner delay times [typical Wigner delay times] vs. the logarithm of N , see Eq. (24) [Eq. (25)]. We concentrate here on the PBRM model with $\beta = 1$ only.

We start by noticing that if we combine Eqs. (10) and (24) we get a heuristic expression for σ_q as a function of b :

$$\sigma_q \approx \frac{q}{1 + (\alpha_{q+1}b)^{-1}}. \quad (33)$$

In Fig. 5 we compare Eq. (33) to the numerically obtained σ_q as a function of b for some values of q . We observe

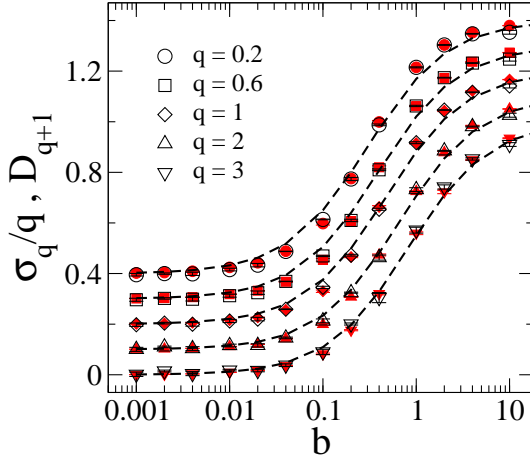


FIG. 5: (Color online) σ_q/q (black open symbols) and D_{q+1} (red full symbols) as a function of b for the PBRM model at criticality with $\beta = 1$. The dashed lines are Eq. (33); we used the values of α_{q+1} reported in Fig. 2(a). For clarity the data for $q < 3$ was displaced upwards.

that Eq. (33) fits reasonably well the numerical σ_q for $q \geq 0.1$. In Fig. 5 we also include independent numerically obtained values of D_q that further verifies the validity of relation (24) [19,22,23].

In Fig. 6 we plot $(q - \sigma_q)[q(\sigma_q + 1)]^{-1}$ as a function of b for the PBRM model at criticality; that is, we verify the validity of Eq. (26). We also plot the analytical prediction for χ given by Eq. (12) and observe good correspondence with the numerical data. In the inset we plot $\sigma_q/(q - \sigma_q)$ as a function of b , see Eq. (29), which for the PBRM model acquires the simple form

$$\sigma_q(q - \sigma_q)^{-1} \approx \alpha_{q+1}b. \quad (34)$$

Finally, in Fig. 7 we show σ_τ as a function of b for the PBRM model at criticality with $\beta = 1$. To test the validity of Eq. (27) we compare σ_τ with the numerically obtained D_1 and with the theoretical prediction for $1 - \chi$. We again observe good correspondence.

IV. OTHER CRITICAL ENSEMBLES

Remember that relations (14-18) were obtained from the combination of Eqs. (10) and (11). That is, relations (14-18) are expected to work in particular for the PBRM model at criticality. However, Eqs. (14) reproduce Eqs. (6) and (7), which were shown to be valid for the PBRM model but also for other critical ensembles.⁴ Therefore the question arises to which extent relations (14-18) are valid for critical ensembles different to the PBRM model. So, in the following we verify the validity of Eqs. (14-18) for other critical ensembles.

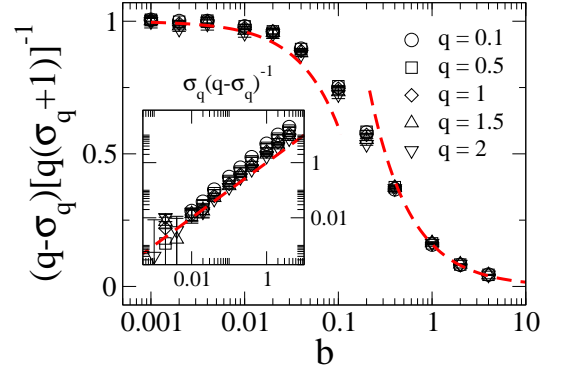


FIG. 6: (Color online) $(q - \sigma_q)[q(\sigma_q + 1)]^{-1}$ [see Eq. (26)] as a function of b for the PBRM model at criticality with $\beta = 1$. The red dashed lines are the analytical prediction for χ given by Eq. (12). Inset: $\sigma_q(q - \sigma_q)^{-1}$ as a function of b ; see Eq. (34). The red dashed line equal to b is plotted to guide the eye.

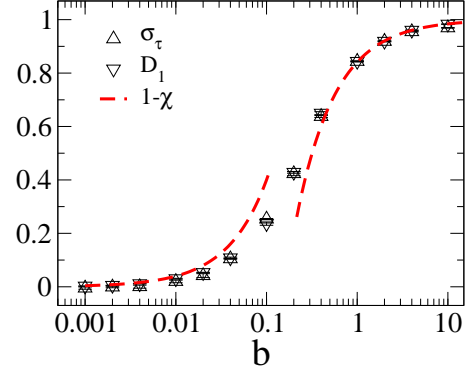


FIG. 7: (Color online) σ_τ as a function of b for the PBRM model at criticality with $\beta = 1$. We include the numerically obtained D_1 , see Eq. (25). The red dashed lines are $1 - \chi$, see Eq. (27).

A. Calogero-Moser ensembles

The Calogero-Moser (CM) N -particle systems yield three ensembles of $N \times N$ Hermitian matrices of the form^{5,6}

$$H_{mn} = p_m \delta_{mn} + i g (1 - \delta_{mn}) V(m - n), \quad (35)$$

where p_m are independent Gaussian random variables with zero mean and unit variance, and $V(m - n)$ is one of the three following functions

$$\frac{1}{m - n}, \quad \frac{1}{N \sinh[(m - n)/N]}, \quad \frac{1}{N \sin[(m - n)/N]}.$$

These ensembles were denoted as^{5,6} CMR, CMH, and CMT, respectively.

In Figs. 8 and 9 we plot D_q as a function of q for the CMR and CMT ensembles, respectively, for several values of g . To have an independent verification of our predictions, the data reported in these figures was taken

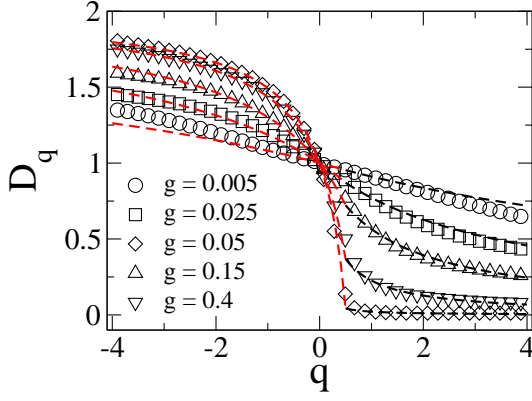


FIG. 8: (Color online) D_q as a function of q for the CMR ensemble for several values of g . The data was taken from Ref. [6]. Black [Red] dashed lines are Eq. (15) [Eq. (21)]. We used $D_1 = 0.911, 0.771, 0.565, 0.226$, and 0.0216 .

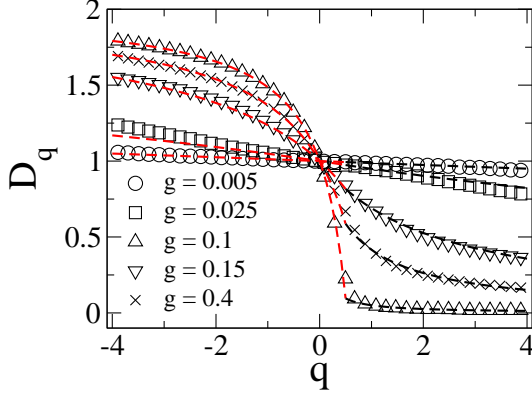


FIG. 9: (Color online) D_q as a function of q for the CMT ensemble for several values of g . The data was taken from Ref. [6]. Black [Red] dashed lines are Eq. (15) [Eq. (21)]. We used $D_1 = 0.987, 0.949, 0.836, 0.416$ and 0.0502 .

from Ref. [6]. We compare the numerical data with our equations for D_q with $q < 1/2$, Eq. (21), and $q > 1/2$, Eq. (15); using as input, values of D_1 obtained by the interpolation of the curves D_q vs. q . We observe for both ensembles that our predictions reproduce reasonably well the numerical data.

B. The Ruijsenaars-Schneider Ensemble and Intermediate quantum maps

The Ruijsenaars-Schneider Ensemble (RSE) proposed in [49] is defined as matrices of the form

$$H_{mn} = \exp(i\Phi_m) \frac{1 - \exp(2\pi i g)}{N[1 - \exp(2\pi i(m - n + g)/N)]}, \quad (36)$$

where $1 \leq m \leq n$, Φ_m are independent random phases distributed between 0 and 2π , and g is a free parameter independent on N .

Now, in Fig. 10 we present D_q as a function of q for the RSE for several values of g . The data was also taken from Ref. [6]. As for the CM ensembles, here we observe that our predictions reproduce reasonably well the numerical data for both D_q with $q < 1/2$ and $q > 1/2$. The values of D_1 we used as input in Eqs. (21) and (15) were obtained by the interpolation of the curves D_q vs. q .

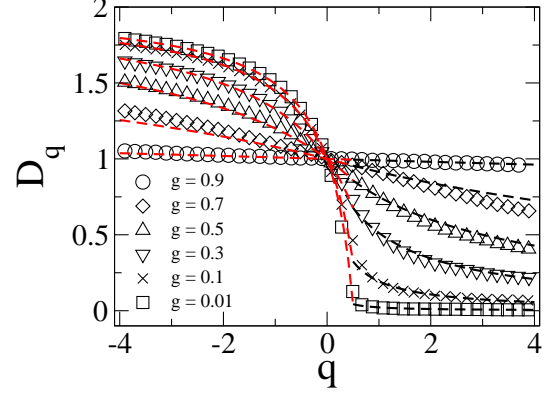


FIG. 10: (Color online) D_q as a function of q for the RSE for several values of g . The data was taken from Ref. [6]. Black [Red] dashed lines are Eq. (15) [Eq. (21)]. We used $D_1 = 0.99, 0.914, 0.75, 0.512, 0.193$, and 0.022 .

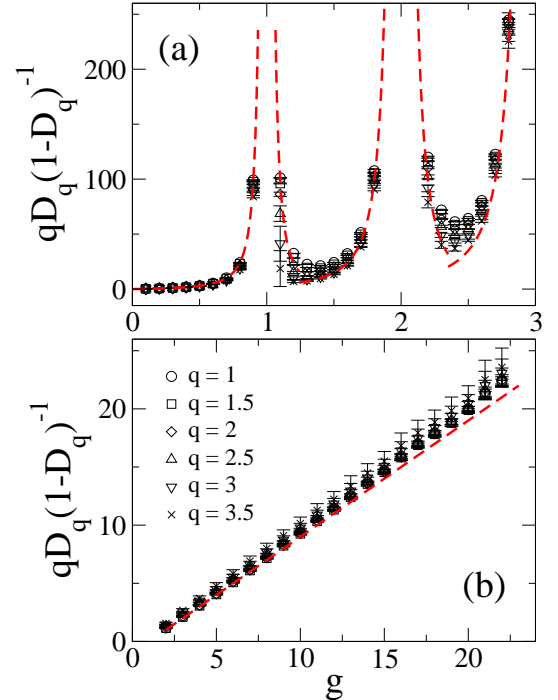


FIG. 11: (Color online) $qD_q(1 - D_q)^{-1}$ as a function of g for (a) the RSE and (b) IQM for several values of q . Red lines are (a) Eqs. (40-41) and (b) Eq. (42). The multifractal dimensions D_q reported in this figure were computed using the same matrix sizes and ensemble realizations as for the PBRM model.

In fact, in Ref. [18] we have already successfully tested the predictions given by Eq. (18) on the RSE. There, we even formulated heuristic expressions for D_q :

$$D_q \approx [1 - (g - 1)^2] [1 + (q - 1)(g - 1)^2]^{-1} \quad (37)$$

and

$$D_q \approx [k^2 - (g - k)^2] [k^2 + (q - 1)(g - k)^2]^{-1} . \quad (38)$$

for $0 < g < 1$ and $|g - k| \ll 1$ with $k \geq 2$, respectively. We got the expressions above by substituting $\chi \sim (g - 1)^2$ and $\chi \sim (g - k)^2/k^2$ [or $D_1 \approx 1 - (g - 1)^2$ and $D_1 \approx 1 - (g - k)^2/k^2$], obtained analytically in Ref. [4], into Eq. (14) [or Eq. (15)].

Also, in Ref. [18] we tested some of our predictions for a variant of the RSE, introduced in [50], with the name of intermediate quantum maps (IQM) model; see also [51]. In this model the parameter g of the RSE is replaced by cN/g with $cN = \pm 1 \bmod g$, being g the parameter of the IQM model. For the IQM model we substituted $\chi \approx 1/g$ or $D_1 \approx 1 - 1/g$, analytical expressions reported in [50], into Eq. (14) or (15), respectively, to get

$$D_q \approx (g - 1)(g + q - 1)^{-1} . \quad (39)$$

Here we just want to add that by the use of Eqs. (37) and (38), for the RSE, and Eq. (39), for the IQM model, we can demonstrate the independence of $qD_q(1 - D_q)^{-1}$ on q (already shown for the PBRM model in Fig. 3). In fact, by substituting the above-mentioned expressions into Eq. (16) we get

$$qD_q(1 - D_q)^{-1} \approx (g - 1)^{-2} - 1 , \quad (40)$$

and

$$qD_q(1 - D_q)^{-1} \approx k^2(g - k)^{-2} - 1 , \quad (41)$$

for $0 < g < 1$ and $|g - k| \ll 1$ with $k \geq 2$, respectively, for the RSE; and

$$qD_q(1 - D_q)^{-1} \approx g - 1 , \quad (42)$$

for the IQM model. Then, In Fig. 11 we plot $qD_q(1 - D_q)^{-1}$ for the RSE and the IQM model for several values of q . We also include the equations given above in red dashed lines. We observe a rather good correspondence between numerical data and Eqs. (40-42).

C. Higher dimensional models

The generalization of Eqs. (14-18) to higher dimensional systems ($d > 1$) can be done if D_q is replaced by D_q/d in Eqs. (14-18). Below we explore the applicability of our results to the Quantum Hall transition in $d = 2$ and the Anderson transition in $d = 3$.

In Fig. 12(a) we plot D_q as a function of q for the Quantum Hall transition (QHT). The data for D_q was

taken from [52]. We also include the prediction for D_q given by Eq. (15) (where D_q has been replaced by $D_q/2$) using $D_1 = 1.7405 \pm 0.0002$ [52]. We observe that the prediction of Eq. (15) is a reasonably good approximation for D_q in the interval $0 < q < 1.2$.

In Fig. 12(b) we plot D_q and D_q^{typ} as a function of q , for the 3d Anderson model at criticality, together with the prediction for D_q given by Eq. (15) (where D_q has been replaced by $D_q/3$) using $D_1 = 1.93 \pm 0.01$ [53]. The data for D_q and D_q^{typ} was taken from [54]. The multifractal dimensions D_q^{typ} were extracted from the scaling of the *typical* participation numbers $\mathcal{I}_q^{\text{typ}} \equiv \exp \langle \ln \mathcal{I}_q \rangle$ with the system size N , from the relation $\mathcal{I}_q^{\text{typ}} \sim N^{-(q-1)D_q^{\text{typ}}}$.

We observe that the prediction of Eq. (15) is reasonably good for D_q^{typ} with $0 < q < 4$. In contrast, Eq. (15) do not reproduce the numerical D_q when $q > 1$. We have also substituted $D_1 = 1.97 \pm 0.002$ (obtained from the interpolation of the D_q data from [54]) into Eq. (15) but the resulting D_q curve is very similar to that with $D_1 = 1.93 \pm 0.01$, so we do not show it in Fig. 12.

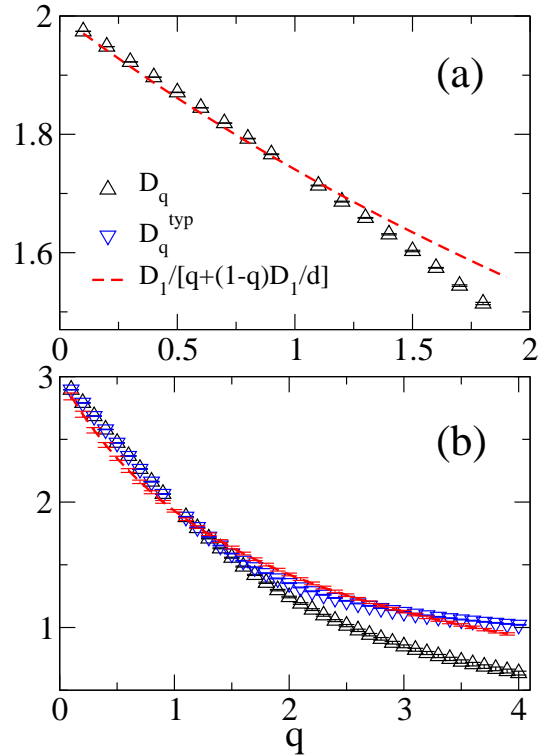


FIG. 12: (Color online) (a) D_q as a function of q for the QHT. The red dashed line is the prediction for D_q given by Eq. (15) using $D_1 = 1.7405 \pm 0.0002$ [52]. The numerical data for D_q was taken from [52]. (b) D_q and D_q^{typ} as a function of q for the 3d Anderson model at criticality. The red dashed line is the prediction for D_q given by Eq. (15) using $D_1 = 1.93 \pm 0.01$ [53]. The numerical data for D_q and D_q^{typ} was taken from [54].

V. APPLICABILITY TO A DETERMINISTIC MODEL

In the previous section we have verified that relations (14-18) are valid for critical random matrix ensembles in 1d and also, to some extent, to higher dimensional models at criticality. The common feature in the systems used above is the presence of multifractal eigenstates. However, note that not only disordered models produce them. It is well known that deterministic models having self-similar potentials also possess multifractal eigenstates, see for example [55,56]. Below, we test the applicability of our expressions relating multifractal exponents now to the multifractal eigenstates of a tight-binding model having a self-similar potential.

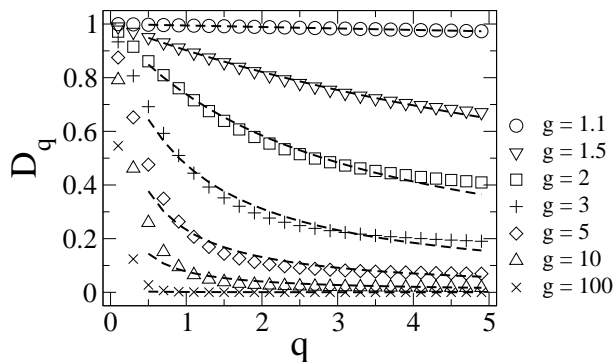


FIG. 13: D_q , computed from Eq. (43), as a function of q for the Fibonacci lattice for several values of g . The dashed line is the prediction for D_q given by Eq. (15) with D_1 calculated from Eq. (44).

A. Off-diagonal one dimensional Fibonacci lattice

According to Fujiwara *et al.* [55] the multifractal spectrum of a one-dimensional Fibonacci sequence can be represented by the inflation rule $T_{n+1} = T_n T_{n-1}$, where $T_1 = A$ and $T_2 = AB$, so $T_3 = ABA$ and so on. In this case the Schrödinger equation

$$t_{j+1}\psi_{j+1} + t_{j-1}\psi_{j-1} = E\psi_j$$

has a multifractal solution at the bandcenter, $E = 0$. Then, by defining the parameter $g = t_{AB}/t_{AA}$ the generalized dimensions of the eigenstates take the form⁵⁵

$$D_q = (3 \ln \sigma)^{-1} (q-1)^{-1} \ln [\lambda^q(g^2)/\lambda(g^{2q})] , \quad (43)$$

where $\sigma = (\sqrt{5} + 1)/2$ is the golden mean, and $\lambda(x) = (2x)^{-1}[(x+1)^2 + \sqrt{(x+1)^4 + 4x^2}]$. Hence, the information dimension and the correlation dimension read as

$$D_1 = (3 \ln \sigma)^{-1} [\ln \lambda(g^2) - g^2 \ln g^2 \lambda'(g^2)/\lambda(g^2)] \quad (44)$$

and

$$D_2 = (3 \ln \sigma)^{-1} [2 \ln \lambda(g^2) - \ln \lambda(g^4)] , \quad (45)$$

respectively.

Then, in Fig. 13 we plot D_q , computed from Eq. (43), as a function of q for the Fibonacci lattice for several values of g . The dashed line is the prediction for D_q given by Eq. (15) with D_1 calculated from Eq. (44). Again, as for the eigenstates of disordered models, here we observe that Eq. (15) reproduces rather well the multifractal dimensions of the eigenstates of the Fibonacci lattice, mainly in the range of $1 < q < 4$.

VI. CONCLUSIONS

In this paper we propose heuristic relations on one hand between the generalized multifractal dimensions, D_q and $D_{q'}$, for a relatively wide range of the parameter q , and on the other hand between these dimensions and the level compressibility χ . As a result we find a general framework embracing an earlier result¹⁰ and a recent one.⁴ Our proposed relations have been corroborated by numerical simulations on various random matrix ensembles, as well as on a deterministic model having a self-similar potential, whose eigenstates have multifractal properties. Of course the analytical relations and the numerical simulations set limitations on the validity for a certain range of the parameter q . Therefore our results are obviously approximate. Hence they call for further theoretical as well as numerical investigations.

Moreover, since our relations between the generalized dimensions and the level compressibility allowed us to state a clear link between the spectral and scattering properties of disordered systems at the metal-insulator transition, it may be interesting to explore the consequences of our results on the quantities characterizing the dynamical properties of critical random matrix ensembles; which have been the focus of very recent investigations.⁵⁷⁻⁵⁹

We believe that our results may find applications in several recently studied models characterized by multifractal eigenstates; such as deterministic self-similar potentials,^{55,56} quantum spin chains,⁶⁰ Dirac fermions in the presence of random magnetic fields,⁶¹ and other critical random matrix ensembles.^{4,62}

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